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2007 J. Phys. A: Math. Theor. 40 F631

(http://iopscience.iop.org/1751-8121/40/27/F08)

IOP FTC

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) F631–F637 doi:10.1088/1751-8113/40/27/F08

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On the Vlasov approach to tokamak equilibria with flow

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Received 18 April 2007, in final form 31 May 2007
Published 20 June 2007
Online at stacks.iop.org/JPhysA/40/F631

Abstract

A previous proof of non-existence of tokamak equilibria with purely poloidal flow within macroscopic theory (Throumoulopoulos *et al* 2006 *Phys. Plasmas* **13** 122501) motivated this microscopic analysis near the magnetic axis for toroidal and 'straight' tokamak plasmas. Despite the new exact solutions of Vlasov's equation on the magnetic axis found here, the structure of macroscopic flows remains elusive. However, the treatment within a toroidal system of orthogonal coordinates and comparison with the straight case, which reveals the importance of toroidicity, is encouraging.

PACS numbers: 51.10.+y, 52.65.Ff, 52.55.−s, 52.55.Fa

1. Introduction

There are a number of publications on kinetic steady states with flow in the framework of Vlasov [1–6], gyrokinetic [7, 8] and collissional [9] theories. For a tokamak plasma collisionless steady state solutions are usually based on the two known constants of motion: the energy and the angular momentum conjugate to the ignorable coordinate. For this reason, the majority of the above-referenced papers deals with toroidal flows [2–4, 6, 7]. To construct axisymmetric equilibria with flows of the arbitrary direction in toroidal geometry, all four constants of motion are generally required and consequently self-consistent solutions of the Maxwell–Vlasov equations. This remains a far from completely solved problem.

In the framework of the macroscopic (magnetohydrodynamic) theory, some time ago (see [10, 11]), it was possible to prove non-existence of tokamak equilibria with purely poloidal incompressible flow. Recently, an extension to compressible plasmas appeared in [12] including the Hall term and pressure anisotropy. The proof for the incompressible case given in [10, 11] was global, while the recent proof [12] is limited to the neighbouring of the magnetic axis through a kind of Mercier expansion.

This last result motivated the idea to extend, in the present study, the analysis to Vlasov–Maxwell equations. The major part of the study concerns considerations near the magnetic axis. An important ingredient is to write the Vlasov equation in cylindrical coordinates in a tokamak geometry, which simplifies the subsequent analysis. We use for this purpose the calculation done in an old ICTP report [13] where the Vlasov equation is written in arbitrary orthogonal coordinates.

In section 2 the expression of the Vlasov equation is obtained in toroidal geometry. In section 3 the ODEs of the characteristics are derived and certain constants of motion are constructed on the magnetic axis in connection with the question of toroidal and poloidal flows. Section 4 specializes to the respective problem for 'straight tokamaks'. Toroidal and cylindrical equilibria with flows away from the axis are the subjects of sections 5 and 6, respectively. Section 7 is left for discussion and conclusions.

2. Vlasov equation in orthogonal coordinates

As explained in [13] we consider a general system of orthogonal coordinates x^1, x^2, x^3 with the metric $\mathrm{d}s^2 = g_{11}(\mathrm{d}x^1)^2 + g_{22}(\mathrm{d}x^2)^2 + g_{33}(\mathrm{d}x^3)^2$ and unit vectors $\mathbf{e}_i = \frac{\nabla x^i}{|\nabla x^i|}$, where i goes from 1 to 3. The velocity vector of a 'microscopic' fluid element is then projected on the unit vectors $\mathbf{e}_i$ as

$$\mathbf{v} = v^i \mathbf{e}_i, \tag{1}$$

where the components v^i are independent of space variables. It is recalled that the Vlasov equation is an approximation to the N-particles Liouville equation, which replaces the N particles by a continuum. This is sometimes termed 'fluid approximation'. The word microscopic we used has precisely this meaning. It refers to the 'microscopic fluid' with 'microscopic velocity' $\mathbf{v}$. The total derivative of $\mathbf{v}$ is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{E} + \mathbf{v}\times\mathbf{B}, \tag{2}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields consistent with Maxwell equations and the charge to mass ratio $\frac{e}{m}$ is set to 1. Note that, though the velocity components do not have a spatial dependence, the convective derivative in equation (2) in general does not vanish acting on the basis vectors of the coordinate system. Projecting (2) on the unit vectors we obtain

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} = \mathbf{e}_i\cdot(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + \mathbf{e}_i\cdot\mathbf{v}\times\nabla\times\mathbf{v}. \tag{3}$$

Finally, the Vlasov equation in orthogonal coordinates is given by

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \mathbf{e}_i\cdot(\mathbf{E} + \mathbf{v}\times\mathbf{B})\frac{\partial f}{\partial v^i} + (\mathbf{e}_i\cdot\mathbf{v}\times\nabla\times\mathbf{v})\frac{\partial f}{\partial v^i} = 0, \tag{4}$$

where f is a function of the x^i, v^i and time while $\mathbf{v}$ is given by equation (1). For more details see [13]. f stays here for the ion distribution, while the distribution function for the electrons is governed by an equation similar to equation (4).

Let us now consider equation (4) in toroidal geometry by specializing on cylindrical coordinates $x^1 = r, x^2 = \phi, x^3 = z$. The coordinate ϕ corresponds to the toroidal direction and r and z represent the poloidal plane. Then $\nabla\times\mathbf{e}_i = 0$ for $i = 1$ and 3 and $\nabla\times\mathbf{e}_2 = \mathbf{e}_1\times\nabla\phi$. If we replace indices 1, 2, 3 by r, ϕ, z, then we have $\nabla\times\mathbf{v} = v^\phi\mathbf{e}_r\times\nabla\phi$ and

$$\mathbf{v}\times\nabla\times\mathbf{v} = -\frac{v^r v^\phi \mathbf{e}_\phi}{r} + \frac{(v^\phi)^2\mathbf{e}_r}{r}. \tag{5}$$

So the last term of equation (4) becomes $\left[\frac{(v^\phi)^2}{r}\frac{\partial f}{\partial v^r} - \frac{v^r v^\phi}{r}\frac{\partial f}{\partial v^\phi}\right]$. Setting $\mathbf{B} = \mathbf{e}_\phi B^\phi = \mathbf{e}_\phi\frac{I}{r}$ near the axis and $\frac{\partial f}{\partial t} = 0$ for the steady state, equation (4) reads

$$\mathbf{v}\cdot\nabla f + (\mathbf{e}_i\cdot\nabla\Phi)\frac{\partial f}{\partial v^i} - \frac{[v^z I - (v^\phi)^2]}{r}\frac{\partial f}{\partial v^r} + \frac{v^r I}{r}\frac{\partial f}{\partial v^z} - \frac{v^r v^\phi}{r}\frac{\partial f}{\partial v^\phi} = 0. \tag{6}$$

Assuming $\nabla f = \nabla\Phi = 0$ on the axis the final equation to solve is

$$-[v^z I - (v^\phi)^2]\frac{\partial f}{\partial v^r} - v^r v^\phi \frac{\partial f}{\partial v^\phi} + v^r I \frac{\partial f}{\partial v^z} = 0. \tag{7}$$

3. ODEs for characteristics

Let us start with the simpler case $I = 0$, then the characteristics of equation (7) are given by the solution of

$$-\frac{\mathrm{d}v^r}{(v^\phi)^2} = \frac{\mathrm{d}v^\phi}{v^r v^\phi}, \tag{8}$$

whose solution is $(v^r)^2 + (v^\phi)^2 = C$. Since $f = f(C, v^z) = f[((v^r)^2 + (v^\phi)^2), v^z]$ on the axis we obtain for the toroidal flow

$$\int v^\phi f\, \mathrm{d}^3\mathbf{v} = 0, \tag{9}$$

which means zero toroidal flow on the axis.

For $I \neq 0$ the characteristics are given by

$$-\frac{\mathrm{d}v^r}{v^z I - (v^\phi)^2} = -\frac{\mathrm{d}v^\phi}{v^r v^\phi} = \frac{\mathrm{d}v^z}{v^r I}. \tag{10}$$

The last equality delivers $C_1 = v^z + I \ln|v^\phi|$, the second characteristic being the particle energy $C_2 = (v^r)^2 + (v^\phi)^2 + (v^z)^2$. C_1 is 'antisymmetric' in v^z but symmetric in v^ϕ, which leads to

$$\int v^\phi f(C_1, C_2)\, \mathrm{d}^3\mathbf{v} = 0, \qquad \int v^z f(C_1, C_2)\, \mathrm{d}^3\mathbf{v} \neq 0. \tag{11}$$

It means that on the magnetic axis the ϕ-flow is zero while the z-flow is finite. This is not acceptable because in tokamaks the magnetic surfaces are well-defined toroidal nested surfaces which degenerate to a close toroidal line on the magnetic axis. This and the fact that in the framework of the macroscopic theory the velocity shares the same surfaces with the magnetic field (see for example [11]) imply that both the z and r components of the poloidal flow should vanish on the magnetic axis. Because of the closeness and nesting of the poloidal velocity lines around the axis this vanishing must hold irrespective of the inertial frame of reference, especially the one for which the electric field on the axis is zero as assumed before equation (7). In an attempt to overcome this situation we relax the assumption $\nabla f = 0$ on the axis. Reconsideration of the ODEs for characteristics then yields a known additional constant of motion on the axis, i.e. the angular momentum conjugate to the ignorable coordinate ϕ: $r v^\phi = C_3$. The fourth constant of motion is not obtainable analytically, while C_1 and C_2 remain unaffected. The persistence of C_1 gives rise to the question why the above consideration on the axis results in unacceptable poloidal flows thereon. Possible explanations are the following.

(i) The picture we obtained is not complete because of the missing fourth constant of motion.

(ii) The problem relates crucially to the toroidicity because, as will be shown in section 4, poloidal flows are eliminated on the axis by a similar kinetic analysis in plane geometry. The z-flow found here is the Vlasov expression of the ∇B-drift of the particle gyrocentre, which is unavoidable in toroidal geometry but vanishes in plane geometry. Also, it is noted that the toroidicity plays an important role in the proof of the non-existence of tokamak equilibria with purely poloidal flows in the framework of macroscopic theory [12]. The unacceptable z flows on the axis in connection with C_1 is an indication of possible extension of this proof in the framework of kinetic theory. Note that though distribution functions of the form $f(C_2, C_3)$ can lead to purely toroidal flows, they cannot shade light on the question of purely poloidal flows.

(iii) The above analysis does not take into account the macroscopic property of coincidence of the flow surfaces with the magnetic surfaces which may play a special role in toroidal geometry. To possibly recover this result in the context of kinetic theory a self-consistent treatment of Vlasov and Maxwell equations away from the axis is necessary. This remains a tough unsolved problem.

(iv) Collisions may damp unaccepted flows and therefore a collisional kinetic equation may be appropriate instead of the Vlasov equation (see also section 7).

4. 'Straight' tokamaks

The straight tokamaks do have magnetohydrodynamic solutions with purely poloidal flow as known from previous work [14]. For the purpose of a microscopic theory an appropriate coordinate system is the Cartesian one $x^1 = x, x^2 = y, x^3 = z$ so that the toroidal angular coordinate ϕ is replaced by z and the toroidal field I by B^z. It is clarified here that though there is no toroidicity, we keep the terms toroidal and poloidal in connection with the axial direction z and the perpendicular (x, y) plane, respectively. Since $\nabla \times \mathbf{e}_i$ vanishes for all i, the term $\mathbf{v} \times \nabla \times \mathbf{v}$ in equation (4) disappears.

For the steady state with $\nabla\Phi = 0$ but finite B^z and ∇f on the axis, equation (7) is replaced by

$$v^x \frac{\partial f}{\partial x} + v^y \frac{\partial f}{\partial y} + v^y B^z \frac{\partial f}{\partial v^x} - v^x B^z \frac{\partial f}{\partial v^y} = 0, \tag{12}$$

whose characteristics are given by

$$\frac{\mathrm{d}x}{v^x} = \frac{\mathrm{d}y}{v^y} = \frac{\mathrm{d}v^x}{v^y B^z} = -\frac{\mathrm{d}v^y}{v^x B^z}. \tag{13}$$

The solution of equation (13) is $C_1 = (v^x)^2 + (v^y)^2$, $v^x - B^z y = C_2$ and $v^y + B^z x = C_3$ where $(x, y) = (0, 0)$ on the axis. In contrast to the situation in section 3, C_1, C_2 and C_3 are decoupled from the fourth constant of motion $v^z = C_4$ due to the lack of toroidicity. Therefore distribution functions of the form $f = f(C_1, C_2, C_3)$ can lead to purely poloidal flows near the axis irrespective of the cross section shape in consistence with [14]. Also, circular cylindrical equilibria with purely poloidal flows at any radial point are constructed in section 6.

5. Consideration away from axis in toroidal geometry

Since in the presence of toroidicity consideration on the axis does not provide any information on the existence of purely poloidal flows, in this section we consider the problem at an arbitrary spatial point. To this end, a useful system of orthogonal coordinates for axisymmetric equilibria consists of the distance ρ to the circular magnetic axis with radius R, the angle θ around the magnetic axis and the angle ϕ around the axis of symmetry. The coordinates ϕ and θ are associated with the toroidal and poloidal directions, respectively. This system is particularly convenient for treatment near the axis where it is known that the shape of magnetic surfaces is circular or elliptical. The metric is given by

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2 + (R + \rho\cos\theta)^2\,\mathrm{d}\phi^2. \tag{14}$$

From the coefficients of metric (14) we can, similarly to sections 2 and 3, compute the physical components of $\nabla \times \mathbf{e}_i$, $\nabla \times \mathbf{v}$, the magnetic field $\mathbf{B}$ in terms of the components of the vector potential **A** as well as the ∇ operator. Vlasov equation (4) for $\nabla\Phi = 0$ takes now the form

$$v^\rho \frac{\partial f}{\partial \rho} + \frac{v^\theta}{\rho}\frac{\partial f}{\partial \theta} + (v^\theta B^\phi - v^\phi B^\theta)\frac{\partial f}{\partial v^\rho} + (v^\phi B^\rho - v^\rho B^\phi)\frac{\partial f}{\partial v^\theta} + (v^\rho B^\theta - v^\theta B^\rho)\frac{\partial f}{\partial v^\phi}$$
$$+\left(\frac{(v^\theta)^2}{\rho} + \frac{(v^\phi)^2\cos\theta}{R+\rho\cos\theta}\right)\frac{\partial f}{\partial v^\rho} - \left(\frac{(v^\phi)^2\sin\theta}{R+\rho\cos\theta} + \frac{v^\rho v^\theta}{\rho}\right)\frac{\partial f}{\partial v^\theta}$$
$$+ v^\phi\left(-\frac{v^\rho\cos\theta}{R+\rho\cos\theta} + \frac{v^\theta\sin\theta}{R+\rho\cos\theta}\right)\frac{\partial f}{\partial v^\phi} = 0. \tag{15}$$

The ODEs for characteristics are

$$\frac{\mathrm{d}\rho}{v^\rho} = \frac{\rho\,\mathrm{d}\theta}{v^\theta} = \frac{\mathrm{d}v^\rho}{v^\theta B^\phi - v^\phi B^\theta + \frac{(v^\theta)^2}{\rho} + \frac{(v^\phi)^2\cos\theta}{R+\rho\cos\theta}}$$
$$= \frac{\mathrm{d}v^\theta}{v^\phi B^\rho - v^\rho B^\phi - \frac{v^\theta v^\rho}{\rho} - \frac{(v^\phi)^2\sin\theta}{R+\rho\cos\theta}}$$
$$= \frac{\mathrm{d}v^\phi}{v^\rho B^\theta - v^\theta B^\rho - \frac{v^\phi v^\rho\cos\theta}{R+\rho\cos\theta} + \frac{v^\phi v^\theta\sin\theta}{R+\rho\cos\theta}}, \tag{16}$$

with

$$B^\rho = \frac{1}{\rho}\frac{\partial A^\phi}{\partial\theta} - \frac{A^\phi\sin\theta}{R+\rho\cos\theta}, \tag{17}$$

$$B^\theta = -\frac{\partial A^\phi}{\partial\rho} - \frac{A^\phi\cos\theta}{R+\rho\cos\theta}, \tag{18}$$

$$B^\phi = \frac{1}{\rho}\left(\frac{\partial(\rho A^\theta)}{\partial\rho} - \frac{\partial A^\rho}{\partial\theta}\right). \tag{19}$$

6. Cylindrical limit: R infinite, $\frac{\partial f}{\partial\theta} = 0$

In this limit equations (16) for the characteristics can be integrated to deliver the known first integrals

$$C_1 = v^\phi - \int B^\theta\,\mathrm{d}\rho = v^\phi + A^\phi, \tag{20}$$

$$C_2 = (v^\rho)^2 + (v^\theta)^2 + (v^\phi)^2, \tag{21}$$

$$C_3 = \rho\left(v^\theta + \int B^\phi\,\mathrm{d}\rho\right) = \rho(v^\theta + A^\theta). \tag{22}$$

Note that the 'toroidal' coordinate ϕ and the poloidal coordinate θ represent in this limit the axial and azimuthal directions. The general solution of the Vlasov equation is then $f = f(C_1, C_2, C_3)$ permitting general macroscopic flows in axial and azimuthal directions consistent with macroscopic theory [14]. In particular, distribution functions of the forms $f = f(C_1, C_2)$ and $f(C_2, C_3)$ can yield purely toroidal and purely poloidal flows, respectively.

It turns out that for R finite and $\frac{\partial f}{\partial\theta} \neq 0$ only two characteristics out of four can be found explicitly $C_1 = (R+\rho\cos\theta)(v^\phi + A^\phi)$ and C_2 as in (21). We made many trials including the use of Mathematica to find the remaining constants C_3 and C_4 without success even near the magnetic axis in the simplest case of circular magnetic surfaces ($rA^\phi \propto \rho^2$) and $B^\phi = 0$. Since C_1 and C_2 cannot deliver poloidal flows, we expect that the knowledge of C_3 and C_4 may give us a hint or a microscopic proof of non-existence of tokamaks with purely poloidal flow. This hope is consistent with the expected complexity of C_3 and C_4, which may have to depend upon v^θ, v^ϕ and v^ρ, not only upon v^ϕ or v^θ as in (20) and (22), respectively. Recall also the crucial role of toroidicity as already revealed in sections 3 and 4.

7. Discussion and conclusions

A result of section 3 related to the conservation of energy and the constant of motion C_1 on the axis has obliged us to change the assumptions leading from equation (6) to equation (7), i.e. $\nabla f \neq 0$ instead of zero thereon. Then, in the case of straight equilibria of arbitrary cross section complete construction of all four constants of motion implies that purely poloidal flows near the axis are possible. In the presence of toroidicity and $\nabla f \neq 0$ on the axis, however, the special canonical ϕ-momentum solution leads naturally to toroidal flows but not to poloidal flows. For this reason in sections 5 and 6 we have considered the toroidal problem away from the axis. However, for toroidal configurations a comprehensive discussion of the problem cannot be done since the complete set of characteristics of equation (6) or equation (15) is not known. Sections 4, 5 and 6 illustrate this point in establishing a bridge between the toroidal and straight tokamaks, which suggests a more promising elaboration of the problem in the future.

Finally, though we know from section 3 that f must be a function of C_1 and C_2 we could, in addition, choose f to have different values for different signs of, for instance, v^ϕ. A known example of this kind of solutions is the case of BGK waves [15], in which the 'free particles' have different distributions for different signs of their velocities. See also [16] for a quasi-neutral treatment. Though toroidal flows can then be constructed, physical constraints like isotropy of the pressure tensor or constraints on other moments or geometrical symmetries and, ultimately, collisions could exclude such solutions. Again we are led to look for the general solutions of equation (6) or equation (15) with $\nabla f \neq 0$ in order to discuss the structure of the macroscopic flows. As mentioned before, this issue remains an open question.

Acknowledgments

The authors would like to thank Professor Harold Weitzner for useful discussions. Part of this work was conducted during a visit of the author GNT to the Max-Planck-Institut für Plasmaphysik, Garching. The hospitality of that Institute is greatly appreciated. This work was performed within the participation of the University of Ioannina in the Association Euratom-Hellenic Republic, which is supported in part by the European Union and by the General Secretariat of Research and Technology of Greece. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

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